TASI Lecture 1: Introduction To Topological Field Theory

June 12, 2023

1. General Plan

These lectures are meant to be *very elementary* introductions to topological field theory and differential cohomology.

Lectures 1+2 concern TFT

Lectures 3+4 concern differential cohomology

2. Basic Picture In TFT: Heuristic Motivation

One central goal of physics is to describe/predict time evolution of quantum systems. This is abstracted in QM/QFT to describing amplitudes.

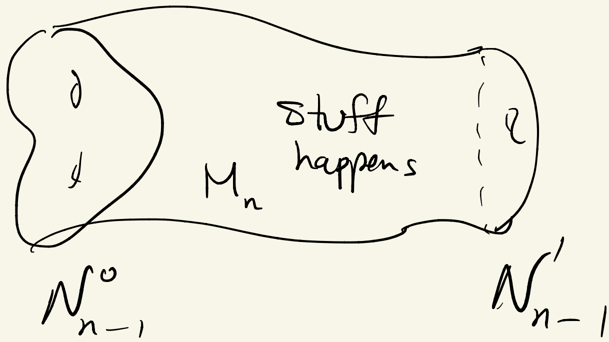
In n spacetime dimensions we might have:

initial spatial manifold N_{n-1}^{0}

final spatial manifold N_{n-1}^{1}

and then there is a spacetime that

interpolates between them



In quantum theory

$$N^0_{n-1} \rightsquigarrow \mathcal{H}_0 : \text{ Hilbert space from which we describe initial states}$$

$$N^1_{n-1} \rightsquigarrow \mathcal{H}_1 : \text{ Hilbert space from which we describe final states}$$

The interpolating history gives a linear map $F: \mathcal{H}_0 \longrightarrow \mathcal{H}_1$

Topological Field Theory (TFT) is meant to capture this very basic idea in a way which expresses locality but eliminates almost all the complications of typical quantum systems.

But in TFT we postulate that only the diffeomorphism class of N^0, N^1, M matters.

But the formulation of TFT motivates a framework: The functorial formulation of field theory for describing general field theories. As such it is a topic of current research, while TFT has a well-developed mathematical theory with rigorous results and is also being further developed.

So, we are going to axiomatize the answers we get from, say, a path integral.

Topological $\Longrightarrow$ no metric dependence, in particular no choice of signature. But we should think of it as axiomatizing Euclidean / Wick-rotated QFT.

To a closed N_{n-1}, $\partial N_{n-1} = \phi$ $\longrightarrow$ $F(N_{n-1})$: A vector space, "the space of states"

Isomorphism class only depends on diffeo class of N_{n-1}. "Topological invariant"

$$(*) \quad F(N_{n-1} \amalg N'_{n-1}) = F(N_{n-1}) \otimes F(N'_{n-1})$$

$\hookrightarrow$ in Q.M. $\mathcal{H}_1, \mathcal{H}_2$ for noninteracting systems then combined system has space of states $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Remarks: 1. $(*)$ Is the beginning of the implementation of locality: LOC1

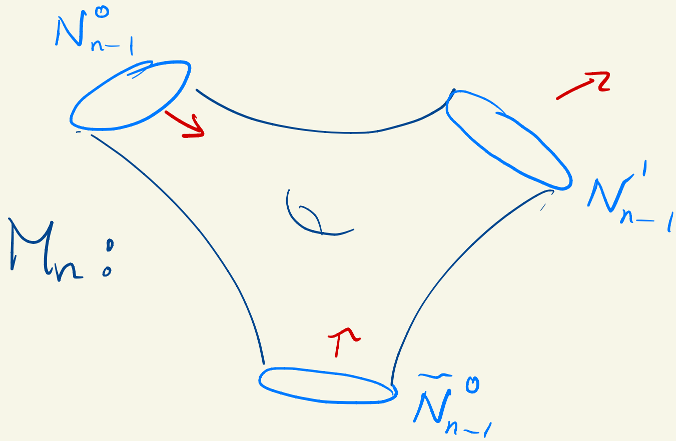
2. Note Well! It follows from $(*)$ that $F(\phi_{n-1}) = \mathbb{C}$

3. Compare with traditional topological invariants:

$$H_k(M \amalg M') = H_k(M) \oplus H_k(M')$$

$\pi_1(M \amalg M')$ Not even defined: need to choose a basepoint.

Now consider an n-manifold with boundary



We want to think of this as a spacetime connecting in- and out- states. We need to choose which spatial slices are "in" and which are "out." e.g. red arrows above indicate in and out.

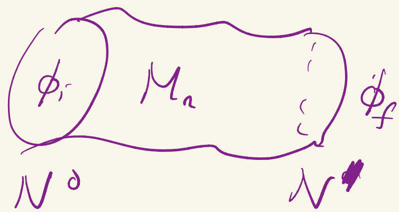
N.B.! We did not assume our manifolds are oriented! We could (and will) consider an analogous story for oriented manifolds, but that is not necessary here.

So, in the above example, our quantum amplitudes will give a linear map:

$$F(M_n) : \mathcal{H}(N^{\circ}_{n-1} \amalg \tilde{N}^{\circ}_{n-1}) \longrightarrow \mathcal{H}(N'_{n-1})$$

The next aspect of locality we wish to axiomatize is gluing. In a QFT the Feynman path integral over field configurations on M_n defines a "propagator" or kernel map on initial and final field configurations:

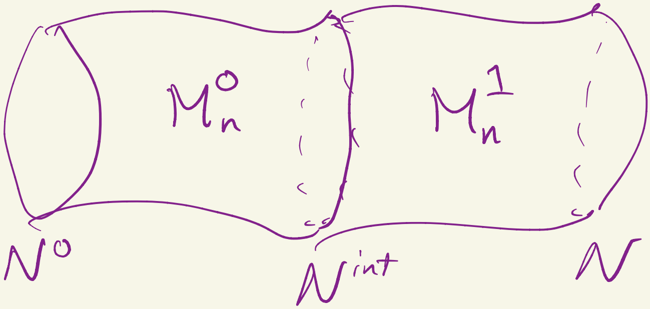
$$K(\phi_f, \phi_i) =$$



Then the amplitude $F(M_n)$ would be expressed as

$$\big(F(M_n)\Psi_i\big)(\phi_f) = \int K(\phi_f, \phi_i)\,\Psi_i(\phi_i)\,d\phi_i$$

But we expect that if we cut M_n along some intermediate $(n-1)$-fold



ϕ_i $\qquad\qquad$ ϕ_f

$$K(\phi_f, \phi_i) = \int \underbrace{K(\phi_f, \phi_{int})}_{M_n^1} \underbrace{K(\phi_{int}, \phi_i)}_{M_n^0} d\phi_{int}$$

This motivates the gluing axiom:

$$F(M_n) = F(M_n^1) \circ F(M_n^0)$$
$$= F(M_n^1 \circ M_n^0)$$

gluing along N^{int}

This is the second aspect of locality we wish to include: LOC2

Now, we might not yet be able to give rigorous mathematical definitions to the most interesting path integrals for quantum field theory — but we can certainly axiomatize certain properties we would definitely want these path integrals to satisfy. The above gluing axiom is an example of such a property.

To put this on a nice and precise mathematical foundation we introduce the idea of "bordism."

3. Bordisms

For much more about bordism theory (with a view to applications in TFT, see: Dan Freed, "Bordism: Old and New"

To save space we will refer to a compact manifold without boundary as a "closed manifold."

Def: Let N^0_{n-1}, N^1_{n-1} be closed manifolds. A **bordism from N^0_{n-1} to N^1_{n-1}** is the following collection of data:

a.) Compact n-manifold with boundary M_n

b.) Decomposition of boundary components into "in" and "out" $\partial M_n = (\partial M_n)^{in} \amalg (\partial M_n)^{out}$

c.) Diffeo's of collar neighborhoods

$$\theta_{in} : N^0_{n-1} \times [0, \varepsilon) \longrightarrow M_n$$

$$N^0_{n-1} \times \{0\} \longrightarrow (\partial M_n)^{in}$$

$$\theta_{out} : N^1_{n-1} \times (1-\varepsilon, 1] \longrightarrow M_n$$

$$N^1_{n-1} \times \{1\} \longrightarrow (\partial M_n)^{out}$$

- A diffeomorphism of bordisms:

$$\begin{tikzcd} N^0_{n-1} \arrow[rr, bend left, "{(M_n, \theta_{in}, \theta_{out})}"{name=U}] \arrow[rr, bend right, "{(M'_n, \theta'_{in}, \theta'_{out})}"'{name=D}] & & N^1_{n-1} \arrow[from=U, to=D, phantom, "\downarrow \psi"] \end{tikzcd}$$

is a diffeo $\psi : M_n \to M'_n$
so that

$$\left.\begin{array}{c} \begin{tikzcd} & M_n \arrow[d, "\psi"] \\ N^0 \times [0, \varepsilon) \arrow[ur, "\theta_{in}"] \arrow[r, "\theta'_{in}"'] & M'_n \end{tikzcd} \\ \\ \begin{tikzcd} & M_n \arrow[d, "\psi"] \\ N^1 \times (1-\varepsilon, 1] \arrow[ur, "\theta_{out}"] \arrow[r, "\theta'_{out}"'] & M'_n \end{tikzcd} \end{array}\right\} \text{commute}$$

- One of the reasons it is useful to incorporate the data of $\theta_{in}, \theta_{out}$ in the definition of a bordism is that it allows us to <u>glue</u> bordisms

$$(M_n, \theta_{in}, \theta_{out}) : N^0 \longrightarrow N^1$$

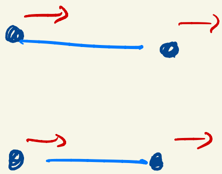
$$(M_n', \theta_{in}', \theta_{out}') : N^1 \longrightarrow N^2$$

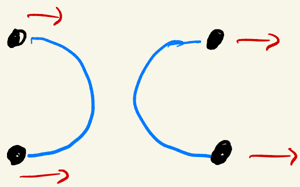
into a single bordism $N^0 \to N^2$.

If we just considered manifolds with boundary we could not account for the possible "twists" in gluing the two bordisms together.

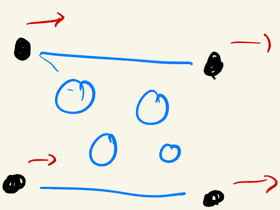
Example: Consider the following 0-manifolds $N^0 = N^1 =$ disjoint union of two points

Question: How many bordisms $: \to :$?
(up to diffeo)





But also



etc. ∞ly many

because we can take disjoint union with bordisms $\emptyset \to \emptyset$, i.e. closed compact manifolds

In TFT we associate to a bordism a linear map $F(\{M_n, \theta_{in}, \theta_{out}\})$ from $F(N^0_{n-1}) \longrightarrow F(N^1_{n-1})$, and postulate that it only depends on the "diffeo equivalence class" of the bordism:

<u>Corollary</u>: $F(N_{n-1})$ is a representation of $\mathrm{Diff}(N_{n-1})$.

<u>Note</u>: The representation factors through to a representation of the mapping class group $\pi_0(\mathrm{Diff}(N_{n-1}))$.

<u>Side remark on maths</u> ① Let G be a topological group. Then the connected component of the identity, G_0, is a normal subgroup (exercise!) and

$$1 \to G_0 \longrightarrow G \longrightarrow \pi_0(G) \to 1$$

So $\pi_0(G) \cong G/G_0$ is a group.

For a general topological space $\mathcal{X}$ $\pi_0(\mathcal{X})$ is not a group.

A good example of a nontrivial $\pi_0(\mathrm{Diff}(N))$ is given by taking $N = T^2 = (\mathbb{R}\oplus\mathbb{R})/\mathbb{Z}\oplus\mathbb{Z}$. The linear tmn. on $\mathbb{R}\oplus\mathbb{R}$

$$\begin{pmatrix} \sigma^1 \\ \sigma^2 \end{pmatrix} \longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \sigma^1 \\ \sigma^2 \end{pmatrix} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{Z})$$

is compatible with $\mathbb{Z}\oplus\mathbb{Z}$ group action and descends to ~~an~~ a transformation on the torus. The projection to the quotient group $\pi_0(\mathrm{Diff}(T^2))$ is nontrivial because it acts nontrivially on $H_1(T^2,\mathbb{Z})$. Note this example is somewhat atypical since we have presented a $GL(2,\mathbb{Z})$ subgroup of $\mathrm{Diff}(T^2)$ rather than as a quotient group. In general we could not do that.

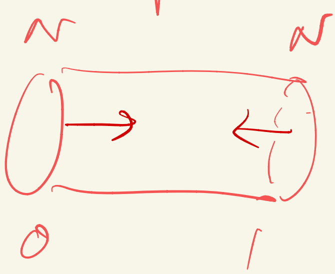
② Bordism Groups

Using the data of $\Theta_{in}, \Theta_{out}$ we can glue together bordisms

$$\begin{tikzcd} N^0 \arrow[r, "{(M,\theta,\theta)}"] \arrow[rr, bend right, "{(M,\theta,\theta)\circ(M,\theta,\theta)}"'] & N^1 \arrow[r, "{(M,\theta,\theta)}"] & N^2 \end{tikzcd}$$

to prove that bordism is an equivalence relation on $(n-1)$-manifolds.

Under disjoint union we form an Abelian group which is 2-torsion because



Shows $[N] \sqcup [N] \cong [\phi_{n-1}]$

Note that $\Omega_1 = \{0\}$ is the trivial group, because every circle bounds a disk.

But $[\mathbb{RP}^2] \in \Omega_2$ is nontrivial: if $\exists\ \partial M_3 = \mathbb{RP}^2$ then gluing $M_3 \cup_{\mathbb{RP}^2} M_3$ = closed 3-fold with Euler characteristic $2\chi(M_3) - 1$, but this must vanish.

In fact, by the classification thm for surfaces $\Omega_2 \cong \mathbb{Z}/2\mathbb{Z}$.

Bordism groups play an important role in the application of TFT to the mathematical theory of topological phases of matter.

Returning to TFT, the rules are:

$$F : \left\{ \begin{array}{l} \text{cpt } (n-1)\text{-folds} \\ \cdot\ \partial N = \phi \end{array} \right\} \longrightarrow \text{Vector spaces}$$

(over $\mathbb{C}$, in these lectures)

$$F(N \amalg N') \cong F(N) \otimes F(N')$$

$$\left(\Rightarrow F(\phi_{n-1}) = \mathbb{C} \right)$$

$$F : \begin{pmatrix} \text{Bordism} \\ M : N^0 \longrightarrow N^1 \end{pmatrix} \longrightarrow \overset{\text{linear tsm}}{F(M_n) \in \mathrm{Hom}_{v.s.}(F(N^0), F(N^1))}$$

such that

$$F(M_n' \circ M_n) = F(M_n') \circ F(M_n)$$

(Note: For disjoint bordisms $F(M_n \amalg M_n') = F(M_n) \otimes F(M_n')$)

<u>Corollaries</u>:

1.) Since $\mathrm{Hom}(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}$ canonically:

(Every lin. tsm. $\mathbb{C} \to \mathbb{C}$ is of the form $T(z) = z_0 z$ so $T \mapsto z_0 := T(1)$)

$\Rightarrow$ For M_n cpt, without bdry. $M_n : \phi_{n-1} \longrightarrow \phi_{n-1}$

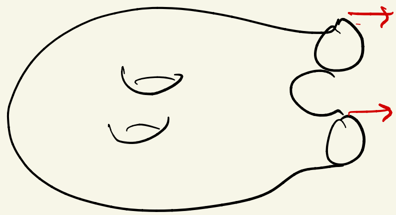
$F(M_n) \in \mathbb{C}$ "partition function"

2.) Consider a bordism

$$M_n : \phi_{n-1} \longrightarrow N_{n-1}$$

Then $F(M_n) \in \mathrm{Hom}\left(\mathbb{C}, F(N_{n-1})\right)$

$$\underline{1} \longmapsto \text{Vector}$$



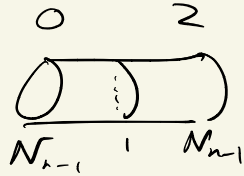
$\Longrightarrow$ Vector in the outgoing Statespace.

<u>Remark:</u> The vector might well be the <u>Zero-vector</u>. In QM a (pure) state is represented by a rank 1 projection operator. If $\psi \in \mathrm{Im}\, P$ and ψ is NONZERO

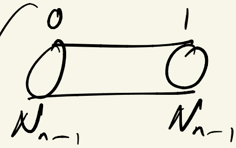
$$P = \frac{|\psi\rangle\langle\psi|}{\|\psi\|^2}.$$

So the term "Statespace" is inaccurate.

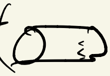
3.) Consider any cylinder $M_n = N_{n-1} \times [0,1]$
with $\theta_{in} = \theta_{out}$ = identity.

$$F(M_n) \circ F(M_n) = F\Big($$ 

$$\Big)$$

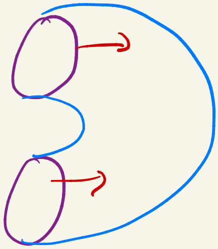
$\underline{\underline{\text{topological!!}}}$ $F\Big($ 

$\Big)$

$\Longrightarrow F(M_n) \in \mathrm{Hom}(F(N_{n-1}), F(N_{n-1}))$
is a projector: All amplitudes are
zero on $\mathrm{Ker}(F(M_n)) \Longrightarrow$ Assume WLOG
$F($  $)$ = Idenity.

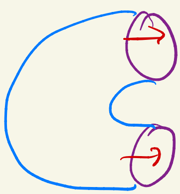
4.) "Dualizability" N cpt. w/out bdry

set $F(N) := V$

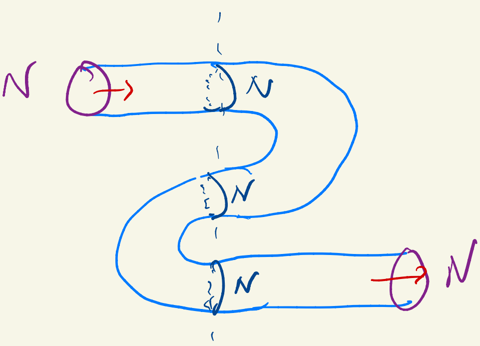
$F\Big($  $\Big) : V \otimes V \longrightarrow \mathbb{C}$

Diff action $\Longrightarrow$ Symmetric bilinear form

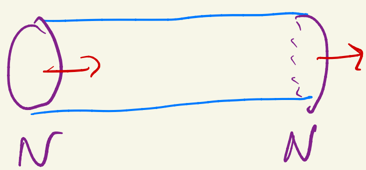
$$V \otimes V \xrightarrow[b]{} \mathbb{C}$$

$F\Big($  $\Big) : \mathbb{C} \xrightarrow[\tilde{b}]{} V \otimes V$

Now consider the "S-diagram"



$=$



This gives a map (recall $V = F(N)$):

$$V \xrightarrow{\mathrm{Id}\otimes\tilde{b}} V \otimes (V \otimes V) \cong (V\otimes V)\otimes V \xrightarrow{b\otimes \mathrm{Id}} V$$

The composition must be the identity. This implies that b is nondegenerate and $V = F(N)$ is finite dimensional.

Pf: Choose a basis: $b(v_i, v_j) = b_{ij}$

$$\tilde{b}(1) = \tilde{b}^{ij}\, v_i \otimes v_j$$

We learn that $\tilde{b}^{ij} b_{jk} = \delta^i{}_k \Longrightarrow$

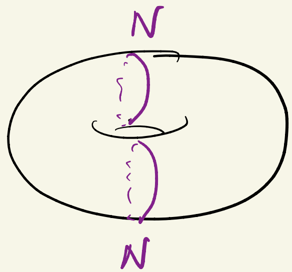
b_{ij} is invertible. If V were ∞ dim'l we could define a H.S. structure declaring $\{v_i\}$ to be O.N. We would want

$b_{ij}\, \check{v}_i \otimes \check{v}_j$ and $\tilde{b}^{ij}\, v_i \otimes v_j$ to be normalizable

This is not possible.

5. It follows that

$$F(N \times S^1) = \dim_{\mathbb{C}}(F(N))$$

$F\Big($  $\Big) = \tilde{b}^{ij} b_{ji} = \delta^i_i$

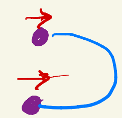
<u>Note:</u> In many discussions of QFT the overall normalization of the path integral gets no respect. This is not the case in TFT where the overall normalization has a definite meaning and for some manifolds is even quantized. One of the many applications is to produce topological invariants:

$F(M) =$ topological invariant, often an <u>enumerative invariant</u>

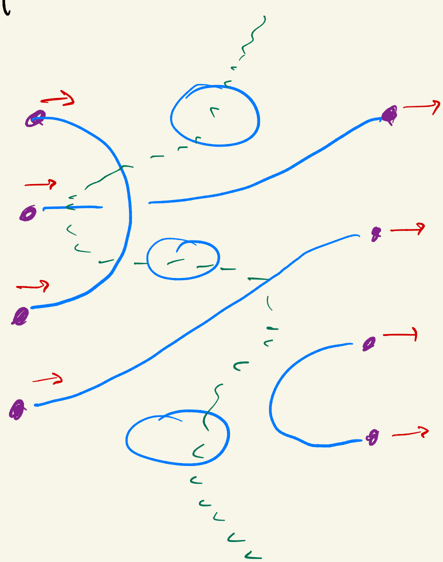
Here the normalization is crucial! We are counting (curves, instantons, monopoles, ...)

4. Example: $n = 1$

$n=1$: $\exists !$ connected 0-dimensional mfld the humble pt. So all state spaces follow from $F(pt) = V$ a f.d. vector space. Then we have nondeg. symmetric form

$F\Big($  $\Big) : V \otimes V \longrightarrow \mathbb{C}$

That's all! We now have the basic data to compute any amplitude we like, such as:



$$V^{\otimes 4} \longrightarrow V^{\otimes 4}$$

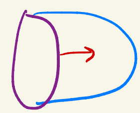
Nontrivial fact: No matter how we cut along intermediate channels we'll get the same result.

5. Example 2: Oriented $n=2$ Theory

$n=2$. To avoid complications with classification of unoriented surfaces we work with oriented bordisms.

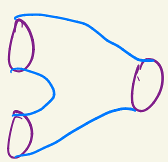
$\exists$ Only 1 connected 1-manifold w/out bdry: $F(S^1) = V$

Two algebraic structures:

Canonical bordism $S^1 \to \phi$, 

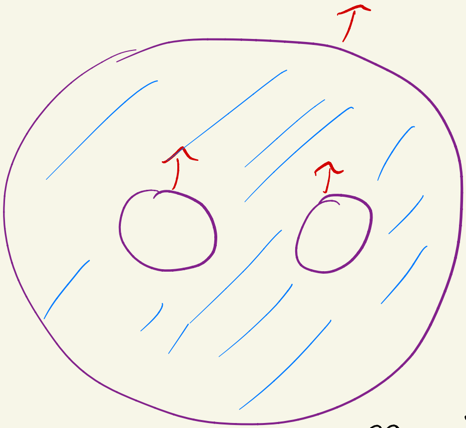
the disk $=$ $\theta: V \to \mathbb{C}$

Multiplication: Pair of pants

 $m: V \otimes V \longrightarrow V$

A useful way to look at it:

Disks within disks:



<u>Corallery 1</u>: $m: V\otimes V \longrightarrow V$ is a commutative and associative multiplication

<u>Corallory 2</u>: For <u>any</u> n-dim'l TFT $F(S^{n-1})$ is a commutative + associative algebra.

In case $n=2$ the bilinear form

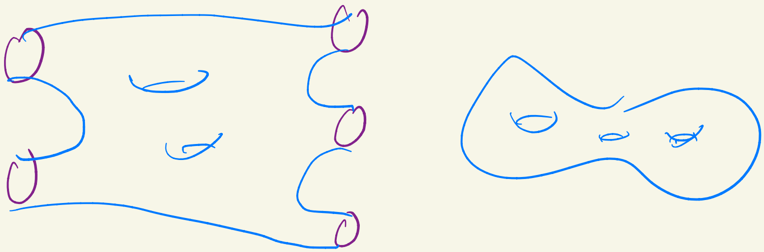
$$b(v_1, v_2) = \theta(v_1 \cdot v_2)$$

is nondegenerate.

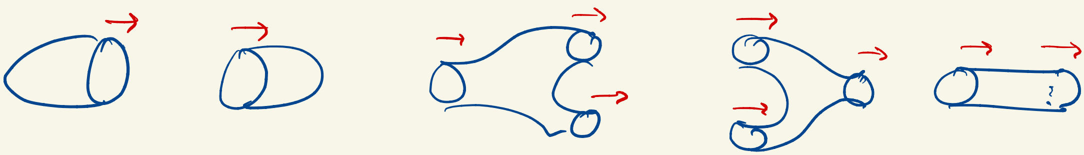
<u>Def</u>: An associative and commutative algebra with $\theta: V \to \mathbb{C}$ s.t. $b(v_1, v_2) = \theta(v_1 v_2)$ is nondegenerate is a <u>Frobenius algebra</u>

Sewing Theorem and Morse Theory

Using the data (V, m, θ) one can compute <u>any</u> amplitude



By cutting into elementary pieces:



The question arises whether two different cuttings into elementary pieces give the same amplitude.

Sewing Theorem: Well-defined amplitudes impose no further algebraic relations on (V, m, θ)

Put differently: To give an $n=2$ dim'l oriented TFT is to specify a Frobenius algebra:

For a long time this was a folk theorem, attributed variously to D. Friedan, R. Dijkgraaf, G. Segal, --

A careful proof is in the appendix of the expository paper of G.M. + G. Segal.

The basic idea is to use a Morse function to give a decomposition of M into level sets. Consider a C^∞ function:

$$f : M \longrightarrow \mathbb{R}$$

"spatial slices" $\quad f^{-1}(t) = N_t \subset M$

$f^{-1}(t)$ will be a nice smooth mfld

Unless t is a critical value.

p: Critical point: $\quad df(p) = 0$

Morse critical point: $\quad \left.\dfrac{\partial^2 f}{\partial x^i \partial x^j}\right|_p$ nondeg.

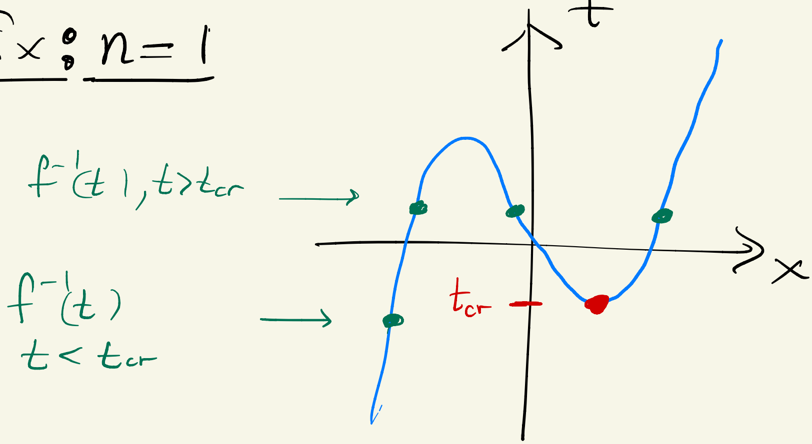
A Morse function on a bordism $M_n : N^0_{n-1} \to N^1_{n-1}$ is excellent if it is constant on N^0, N^1 and the critical points can be ordered so the critical values are

$$c_0 = f(N^0) < c_1 < \cdots < c_N < c_f = f(N^1)$$

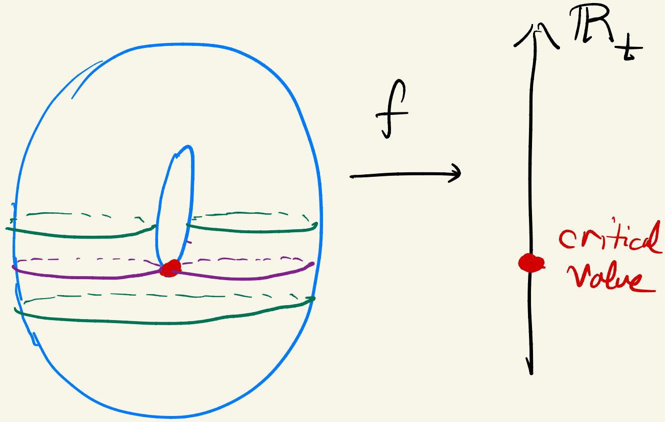
The spatial slices $f^{-1}(t)$ are all diffeomorphic for $c_i < t < c_{i+1}$

But there is topology change as we cross a critical point:

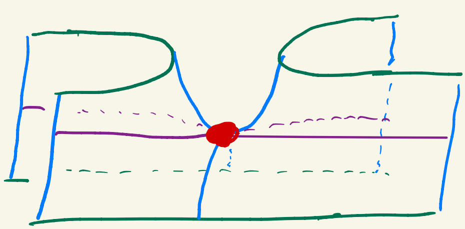
Ex: $n=1$



Ex: $n=2$



Note well, the neighborhood of the critical point looks like this



N.B. This is a "manifold with corners."

Now we can <u>change</u> time-slicings by considering a path of smooth functions f_s which are excellent Morse functions for generic s.

<u>Cerf Theory:</u> In the (Whitney) topology of $C^\infty(M \to \mathbb{R})$ the set of excellent Morse functions is open and dense but <u>disconnected.</u>

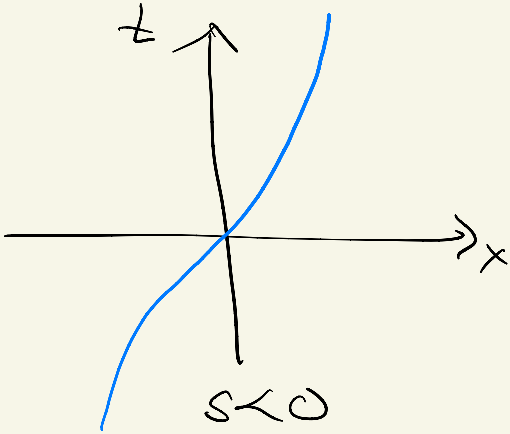
Define a function $f: M_n \longrightarrow \mathbb{R}$ to be "good" if it is Morse everywhere except for one or two critical points, <u>and</u>

- One critical point locally of the form $\pm y^2 + x^3$
- Two critical points have the same value

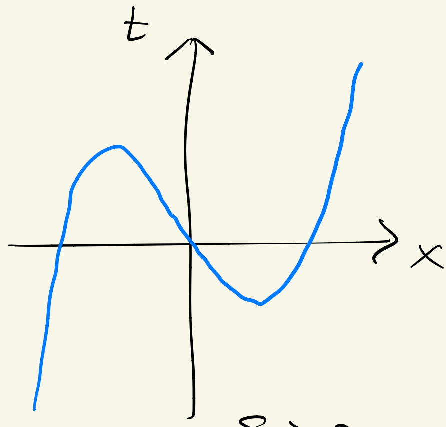
<u>Theorem:</u> The set of excellent and good functions is a <u>connected set</u>. The good but not excellent functions form a real codimension one subset.

So a path of excellent + good functions f_s connecting two time slicings will cross a finite set of critical values $s_1, \ldots, s_k$ where the functions are good but not excellent

<u>Example:</u> $n=1$, $f_s(x) = \dfrac{x^3}{3} - sx$



$s<0$



$s>0$

It now follows from Cerf-Morse theory that any two changes of time slicings are related by some elementary changes. Invariance under those elementary changes is guaranteed by the algebraic axioms of a commutative associative Frobenius algebra. This is how the sewing theorem is proven.

__Semisimplicity__ Choose an ordered basis $\{v_i\}$ for V. Consider the operator L_i defined by left-multiplication by v_i. It has matrix elements:

$$L_i(v_j) = v_i v_j = N_{ij}^{\ k} v_k$$

Commutativity $\Rightarrow [L_i, L_j] = 0$

__If__ the L_i are all diagonalizable we say the algebra V is __semi-simple__

In this case there is a basis of idempotents $\{\varepsilon_i\}$:

$$\varepsilon_i \varepsilon_j = \delta_{ij} \varepsilon_i$$

and the only invariants of the Frobenius algebra are the dimension and the values of the trace $\theta(\varepsilon_i) = \theta_i$

Remarks ① If we view this model as a baby model of string theory with zero-dimensional target space then $\mathfrak{X} = \coprod_i pt_i \quad pt_i \leftrightarrow \varepsilon_i$

and θ_i is the value of the string coupling/dilaton.

② We can also use the $n=1,2$ theories as topological models of quantum gravity. In this context they are useful playgrounds for exploring the role of topology change and "baby universes" in EQG. An important paper on this is

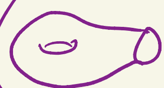
Marolf + Maxfield 2002.08950

with clarifications, generalizations, and further extensions in Banerjee + Moore, 2201.00903

which in turn inspired a general framework for EQG laid out by D. Friedan: 2306.????

<u>Exercise:</u> Suppose V is a semisimple FA

(a.) Show that the state produced by a handle is

$F\Big($  $\Big) = \sum_i \theta_i^{-1} e_i$

(b.) Suppose Σ_g is a connected genus g surface without boundary. Show that

$$F(\Sigma_g) = \sum_i \theta_i^{1-g}$$

(c.) Therefore the vac $\to$ vac, $\phi \to \phi$ bordism given by summing over connected topologies is

$$F_{\text{connected}} = \sum_i \frac{\theta_i}{1-\theta_i} = \sum_i \lambda_i$$

Show that <u>with a suitable weighting of topologies</u> the full vac $\to$ vac amplitudes is

$$F_{\text{vac}\to\text{vac}} = \prod_i e^{\lambda_i}$$

Exercise: Show that $F(\bigcirc)$ defines the unit element for the algebra multiplication in V by illustrating a suitable change of Morse function

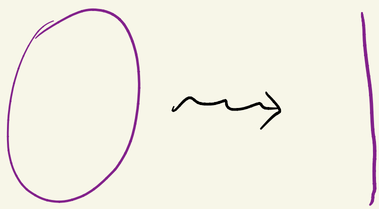
Exercise: Illustrate the change of Morse function that implies the multiplication on V is associative.

Exercise:

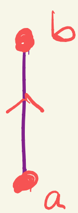
Consider a compact orientable manifold X with all odd Betti numbers $b_i(X) = 0$. Show that the cohomology group $H^*(X, \mathbb{C})$ is a commutative associative Frobenius algebra, but that it is not semisimple. Compute all the amplitudes for $X = \mathbb{CP}^1$.

6. Open–Closed Oriented $n=2$ AND EMERGENCE OF CATEGORIES

If we think of 2d $n=2$ TFT as a model of topological string theory with zero-dimensional target it is natural to ask about the extension to open strings

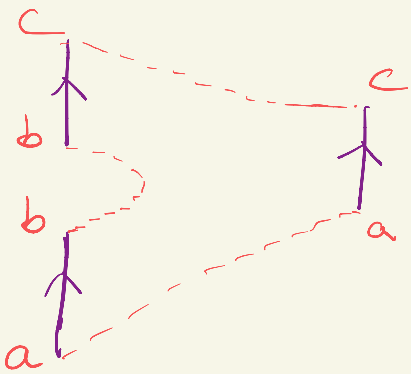
Replace spatial 

Now we need boundary conditions/labels on the end of our open string.
Let's call them $a, b, c, \ldots \in \mathcal{B}_0$

So $F\Big($ 

$\Big) := \mathcal{O}_{ab}$ a vector space of open string states w/ bdry conditions a & b.

Now consider the bordism:



We conclude that:

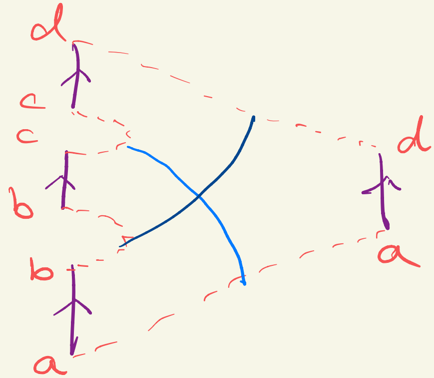
1. $\mathcal{O}_{aa}$ is an associative, but not necessarily commutative algebra.

2. $\mathcal{O}_{ab}$ is a bimodule for $\mathcal{O}_{aa} \times \mathcal{O}_{bb}$

3. There is an associative multiplicati

$$\mathcal{O}_{ab} \times \mathcal{O}_{bc} \longrightarrow \mathcal{O}_{ac}$$ given by the above picture.

The proof of associativity is:



Thus the structure we get is precisely that of a <u>category</u>

<u>Def</u>: A category C is a collection of data (C_0, C_1, p_0, p_1, m) where

(a.) C_0, C_1 are sets

C_0 : "the set of objects" (also denote $C_0 := \mathrm{Obj}(C)$)

C_1 : "the set of morphisms"

(b.) $C_1 \overset{p_1}{\underset{p_0}{\rightrightarrows}} C_0$ codomain / domain maps

Denote $\{f \in C_1 \mid p_1(f) = y \text{ \& } p_0(f) = x\} := C(x,y) := \mathrm{Hom}_C(x,y)$

(c.) Define $C_2 := C_1 \, {}_{p_1}\!\times_{p_0} C_1$

$$= \{(f,g) \mid p_0(f) = p_1(g)\}$$

the set of composable pairs of morphisms

$m: C_2 \longrightarrow C_1$. Denote $m(f,g) := f \circ g$

Satisfying conditions:

(α.) $\forall\, x \in C_0$ $\exists$ morphism $1_x \in C(x,x)$

s.t. $\forall\, f \in \mathrm{Hom}(y,x)$ $\quad 1_x \circ f = f$

$\forall\, g \in \mathrm{Hom}(x,y)$ $\quad g \circ 1_x = g$

(β.) Consider the set of 3 composable morphis.

$$C_3 = \{ (f, g, h) \mid p_0(f) = p_1(g) \;\&\; p_0(g) = p_1(h) \}$$

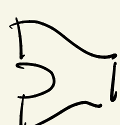
$$\begin{tikzcd} & C_2 \arrow[dr, "m"] & \\ C_3 \arrow[ur, "m\times Id"] \arrow[dr, "Id\times m"'] & & C_1 \\ & C_2 \arrow[ur, "m"'] & \end{tikzcd}$$

i.e. $(f\circ g)\circ h = f\circ(g\circ h)$

$\Rightarrow$ For 2d open-closed TFT: There is a category of boundar conditions:

C_0 = set of boundary conditions a, b, c.

$\mathrm{Hom}(a,b) = \mathcal{O}_{ab}$

m = multiplication 

17. Some Background On Categories

In general it is often useful to think of a category as a directed graph

Objects : Vertices of graph

Morphisms : Oriented edges of graph.

For us, a very important category is the bordism category $\mathrm{Bord}_{\langle n-1,n\rangle}$

Objects: Smooth, closed, $(n-1)$-folds

Morphisms: Bordisms (up to diffeo.)

Composition m: gluing of bordisms.

Exercise: What is the identity morphism?

Another important category for us is

VECT : Objects = f.d. $\mathbb{C}$-vectorspaces

Morphisms = $\mathbb{C}$-linear transformations between v.s.

m = composition of linear maps

With one more idea from category theory we can nicely formalize one key aspect of TFT :

Def: Let $\mathcal{C}, \mathcal{D}$ be two categories. A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a pair of maps

$$F_0 : \mathcal{C}_0 \longrightarrow \mathcal{D}_0$$

$$F_1 : \mathcal{C}_1 \longrightarrow \mathcal{D}_1$$

such that $F_1 : \mathrm{Hom}_{\mathcal{C}}(x, y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F_0(x), F_0(y))$

and $\forall f, g$

either $F_1(f \circ g) = F_1(f) \circ F_1(g)$ (Covariant)

OR $F_1(f \circ g) = F_1(g) \circ F_1(f)$ (Contravariant)

What we've said so far is that the rule F of an n-dim'l TFT is that it is a functor:

$$F : \mathrm{Bord}_{\langle n-1, n\rangle} \longrightarrow \mathrm{VECT}$$

The equation

$$F(f \circ g) = F(f) \circ F(g)$$

captures LOC2.

But what about LOC1?

i.e.

$$F(N \amalg N') = F(N) \otimes F(N')$$

LOC1

To incorporate Γ we need the notion of isomorphism of functors, so we need three more definitions from category theory:

<u>Def</u>: Given categories C, D and two functors
$$\begin{tikzcd} C \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\tau"] & D \end{tikzcd}$$
a <u>natural transformation</u> (a.k.a. "morphism of functors") denoted
$$\tau : F \Rightarrow G$$
is a collection of maps τ_X indexed by $X \in C_0 = \mathrm{Obj}(C)$ such that, for all $x, y \in C_0$ and all $f \in \mathrm{Hom}_C(x,y)$
$$\begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "\tau_X"'] & F(y) \arrow[d, "\tau_Y"] \\ G(x) \arrow[r, "G(f)"] & G(y) \end{tikzcd}$$

Example: The k^{th} integral cohomology is a contravariant functor:

$$H^k_{\mathbb{Z}} : \text{TOP} \longrightarrow \text{AB GROUP}$$

On objects: $H^k_{\mathbb{Z}} : X \longrightarrow H^k(X, \mathbb{Z})$

On morphisms: $H^k_{\mathbb{Z}}(X \xrightarrow{f} Y) =$
(f a continuous map)
$$f^* : H^k(Y, \mathbb{Z}) \to H^k(X, \mathbb{Z})$$

Then the cup product is a $\natural$ tmn between $H^{k_1}_{\mathbb{Z}} \oplus H^{k_2}_{\mathbb{Z}}$ and $H^{k_1+k_2}_{\mathbb{Z}}$

$$H^{k_1}(X, \mathbb{Z}) \oplus H^{k_2}(X, \mathbb{Z}) \xrightarrow[\tau_X = \text{cupproduct}]{} H^{k_1+k_2}(X, \mathbb{Z})$$

Similarly, Steenrod squares are natural transformations.

Exercise For $V \in \mathrm{Obj}(\mathrm{VECT})$

define a functor $F_V : \mathrm{VECT} \longrightarrow \mathrm{VECT}$

by $$F_V(W) := \mathrm{Hom}(V, W) \oplus V$$

$$F_V\left(W_1 \xrightarrow{T} W_2\right) \qquad \mathrm{Hom}(V, W_1) \oplus V$$

$$\xrightarrow{\circ T \oplus \mathrm{Id}} \quad \mathrm{Hom}(V, W_2) \oplus V$$

Show that the evaluation map

$$\tau_W : F_V(W) \longrightarrow W$$

$$A \oplus v \longmapsto A(v)$$

is a natural trnn of F_V to the identity functor. $\mathrm{Id} : \mathrm{VECT} \longrightarrow \mathrm{VECT}$.

Def: An <u>isomorphism</u> of functors $\tau: F_1 \to F_2$ is a natural transformation τ such that there is a natural transformation $\tau': F_2 \to F_1$ with commutative diagrams:

$$\begin{tikzcd} & F_2(X) \arrow[dr, "\tau'_X"] & \\ F_1(X) \arrow[ur, "\tau_X"] \arrow[rr, "Id_{F_1(X)}"'] & & F_1(X) \end{tikzcd} \quad \& \quad \begin{tikzcd} & F_1(X) \arrow[dr, "\tau_X"] & \\ F_2(X) \arrow[ur, "\tau'_X"] \arrow[rr, "Id_{F_2(X)}"'] & & F_2(X) \end{tikzcd}$$

REMARK:

Def: An equivalence of categories C & D is a pair of functors

$F: C \to D$ & $G: D \to C$

with isomorphisms of $F \circ G$ and $G \circ F$ to the identity functors.

Many, many, important results in maths are statements of equivalence of cat's.

Def: A tensor category (a.k.a. "monoidal category")

is a category with a functor

$$\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$

and an isomorphism $\mathcal{A}$ of the functors

$$\begin{tikzcd}
& \mathcal{C} \times \mathcal{C} \arrow[dr, "\otimes"] \arrow[dd, phantom, "\Downarrow \mathcal{A}"] & \\
\mathcal{C} \times \mathcal{C} \times \mathcal{C} \arrow[ur, "\otimes_{12} \times \mathrm{Id}"] \arrow[dr, "\mathrm{Id} \times \otimes_{23}"'] & & \mathcal{C} \\
& \mathcal{C} \times \mathcal{C} \arrow[ur, "\otimes"'] &
\end{tikzcd}$$

$\mathcal{A}$ is known as the associator:

$$\mathcal{A}_{x,y,z} : (x \otimes y) \otimes z \longrightarrow x \otimes (y \otimes z)$$

and it must satisfy the pentagon identity:

$$\begin{tikzcd}[column sep=small]
 & ((X_1 \otimes X_2) \otimes X_3) \otimes X_4 \arrow[r] \arrow[dl] & (X_1 \otimes X_2) \otimes (X_3 \otimes X_4) \arrow[dr] & \\
(X_1 \otimes (X_2 \otimes X_3)) \otimes X_4 \arrow[dr] & & & X_1 \otimes (X_2 \otimes (X_3 \otimes X_4)) \\
 & X_1 \otimes ((X_2 \otimes X_3) \otimes X_4) \arrow[urr] & &
\end{tikzcd}$$

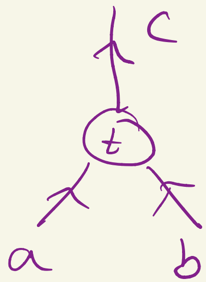
Finally there is an identity object $\mathbb{1}_C \in \mathrm{Obj}(C)$ and natural trans:

$$\imath_L : \mathbb{1}_C \otimes (\cdot) \longrightarrow \mathrm{Id}$$
$$\imath_R : (\cdot) \otimes \mathbb{1}_C \longrightarrow \mathrm{Id}$$

Satisfying some natural compatibility conditions. See EGNO for a complete treatment.

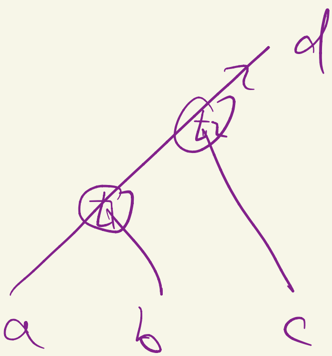
EGNO = Etingof, Gelaki, Nikshych, Ostrik

Remark: Fusion of anyons. A mathematical description of anyons identifies them with objects in a $\otimes$ category. The $\otimes$ is regarded as "fusion of the anyons" and can be pictured as

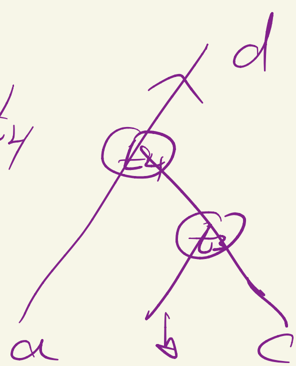


$t \in \mathrm{Hom}(a \otimes b, c)$

The associator is



$$= \sum_{t_3, t_4} F^{t_3 t_4}_{t_1 t_2}$$



<u>Exercise:</u> Write out the pentagon diagram using this notation.

A monoidal $\otimes$-functor between monoidal $\otimes$ categories $F: C \to D$ is a functor that preserves structure in the sense that there are isomorphisms

$$F(X \otimes Y) \xrightarrow[\theta_{X,Y}]{} F(X) \otimes F(Y)$$

$$F(\mathbb{1}_C) \xrightarrow[\eta_C]{} \mathbb{1}_D$$

satisfying some 4 conditions...
(omitted here)

Remark: A <u>braiding</u> is an isomorphism of $\otimes : C \times C \to C$ with

$\otimes \circ \sigma : C \times C \to C$ where

$\sigma : (X, Y) \longrightarrow (Y, X)$ is the exchange functor. This amounts to the data of isomorphisms:

$$\Omega_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$$

<u>Remark</u> In the theory of anyons the anyons are the objects of a tensor catory and $x \otimes y$ is called the fusion of the anyons. $\Omega_{x,y}$ is the braiding.

In these notes we have $\Omega_{y,x} \circ \Omega_{x,y} = Id_{x \otimes y}$.

i.e. we work with <u>symmetric tensor categories</u>.

In general, for anyons, $\Omega_{y,x} \circ \Omega_{x,y}$ is not the identity.

Now VECT is a $\otimes$ category, using $\otimes$ product of vector spaces. The associator is trivial. Also, $\mathrm{Bord}_{\langle n-1,n\rangle}$ is a $\otimes$-category, using disjoint union. They are both symmetric $\otimes$-cat's

Exercise: What is the monoidal unit $\mathbb{1}_C$ in VECT and in $\mathrm{Bord}_{\langle n-1, n\rangle}$?

Def: An n-dim'l TFT is a symmetric $\otimes$-functor

$$F: \mathrm{Bord}_{\langle n-1, n\rangle} \longrightarrow \mathrm{VECT}$$

We'll now give an important example: Finite group gauge theory.

But first, we need some more math ...

8. Some Background On G-Bundles

- For a group G a <u>G-torsor</u> or <u>principal homogeneous space</u> is a set T with a free & transitive G action on T.
- For G a topological group and X a topological space a <u>principal G-bundle over X</u> is a cont. map of topological spaces $\pi : P \longrightarrow X$ such that:

1.) P admits a continuous and free right G-action s.t. $\pi(p \cdot g) = \pi(p)$ and the fibers $\pi^{-1}(x)$ are G-torsors

2.) $\pi : P \to X$ is locally trivial: $\forall x$, $\exists \mathcal{U}_x$

$$\begin{tikzcd}
\pi^{-1}(\mathcal{U}_x) \arrow[rr, "\phi_{\mathcal{U}_x}"] \arrow[dr, "\pi"'] & & \mathcal{U}_x \times G \arrow[dl] \\
& \mathcal{U}_x &
\end{tikzcd}$$

$\phi_{\mathcal{U}_x}$ is G-equivariant.

Key example for us:

Choose $g_0 \in G$ and let $\mathbb{Z}$ act on $\mathbb{R} \times G$ by $n : (x, g) \longrightarrow (x+n, g_0^n g)$

$$P := (\mathbb{R} \times G)/\mathbb{Z} \xrightarrow{\pi} \mathbb{R}/\mathbb{Z} = S^1$$
$$[(x,g)] \longrightarrow [x]$$

Intuitively

Denote this G-bundle / S^1 by P_{g_0}

<u>Def:</u> A bundle map, or morphism of principal G-bundles over X is a fiber-preserving G-equivariant map

$$\begin{tikzcd} P_1 \arrow[rr, "\Phi"] \arrow[dr, "\pi_1"'] & & P_2 \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

One can show:

Every bundle map of principal G-bundles has an inverse bundle map so it defines an isomorph.

Exercise: Show that the bundle map

$$\begin{tikzcd} \mathbb{R}\times G \arrow[rr, "\psi_h"] \arrow[dr] & & \mathbb{R}\times G \arrow[dl] \\ & \mathbb{R} & \end{tikzcd}$$

given by $\psi_h : (x, g) \longmapsto (x, hg)$ induces an isomorphism of bundles over the circle $\psi_h : P_{g_0} \simeq P_{hg_0h^{-1}}$

- Iso. classes of principal G-bundles over S^1 are labelled by conj. classes of elements of G.
- The automorphism group of P_g is

$$Z(g) = \{ h \in G \mid hgh^{-1} = g \}$$

Fact: Let G be a finite group.

Isomorphism classes of principal G-bundles over a topological space X are in 1-1 correspondence with elements of

$$\mathrm{Hom}\big(\pi_1(X, x_0), G\big)/G$$

$\phi \sim \phi'$ if $\exists\, g$ $\phi(\gamma) = g\phi'(\gamma)g^{-1}$ for all $\gamma \in \pi_1(X, x_0)$

Note that setting $X = S^1$ we recover the claim that isom. classes are in 1-1 correspondence with conjugacy classes of G.

9. Finite Group Gauge Theory: Part 1

G-gauge theory for $n = 1$

$$F(S^1) = \text{sum over gauge bundles}$$
$$= \sum_{g \in G} B(P_g)$$

$B(P_g)$ = Boltzmann weight for P_g

This should only depend on the isomorphism class of P_g and hence should be a class function on G. So we write it as a sum over isom. classes.

Being a classfunction we can express the Boltzmann weight as

$$B(P_g) = \frac{\chi_\rho(g)}{|Z(g)|}$$

where χ_ρ is the character for some element ρ in the representation ring of G. $\Longrightarrow$

$$F(S^1) = \sum_{c.c.} \frac{\chi_\rho(g)}{|Z(g)|}$$

Orthogonality relations $\Longrightarrow$ sum projects to identity isotypical component of ρ, WLOG take $\chi_\rho(g) = 1$

Then $F(S^1) = \sum_{c.c.} \frac{1}{|Z(g)|}$

$$= \sum_{P_g} \frac{1}{|G|} = 1$$

So $F(pt) = \mathbb{C}$.

<u>In general:</u>

$$F(N_{n-1}) = \text{Functions}\left[\left\{ \begin{matrix} \text{iso classes of} \\ P \to N_{n-1} \end{matrix} \right\} \to \mathbb{C}\right]$$

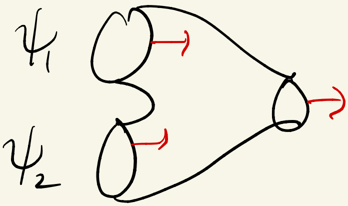
$$F(\underset{\substack{\uparrow \\ \text{closed} \\ \text{w/out bdry}}}{M_n}) = \sum_{[P \to M_n]} \frac{1}{|\mathrm{Aut}\, P|}$$

The discussion of amplitudes is best deferred to after we introduce BG below

For now we just note that:

for $n=2$ we have

$$F(S^1) = \left\{ \begin{array}{c} \text{class functions} \\ \text{on } G \end{array} \right\}$$



Works out to give the convolution product:

$$(\psi_1 * \psi_2)(g) = \sum_{g_1 g_2 = g} \psi_1(g_1)\,\psi_2(g_2)$$

Natural basis are the characters χ_μ in the irreps $\mu \in \mathrm{Irrep}(G)$.

Orthogonality relations for matrix elements of irreps $\Longrightarrow$ $\varepsilon_\mu = \chi_\mu(1)\,\chi_\mu$ is a basis of idempotents

$$\theta(\psi) = \lambda \cdot \psi(1) \text{ defines a}$$

Frobenius structure and applying the above exercise:

$$F(\Sigma_g) = \lambda^{2-2g} \sum_{\substack{\text{irreps } \mu \\ \text{of } G}} (\dim V_\mu)^{2-2g}$$

it can be shown that this is

$$(\lambda |G|)^{2-2g} \frac{\# \mathrm{Hom}(\pi_1(\Sigma_g, x_0), G)}{|G|}$$

so is indeed a sum of the Boltzmann weights $1/|G|$ over isom. classes of G-bundles, up to an "invertible TQFT" (to account for the $(\lambda|G|)^{2-2g}$ factor.)

10. Generalizations: Background Fields

In physics we generally need to endow our spacetimes with geometric structures For example:

- orientation
- (S)pin structure
- Riemannian metric and/or conformal struct.
- Principal G-bundle w/ connection.

In general these structures should satisfy some form of locality:

- They should pull back (or push-forward) under diffeomorphisms
- they should satisfy a sheaf property: If they are defined on open sets U and V and agree on $U \cap V$ then there is a unique extension to $U \cup V$.

For us, these are <u>background</u> <u>fields</u>

The TFT functor gives the answer to the path integral: The dynamical fields have been integrated out.

But the path integral will typically be an interesting function of the background fields.

E.g. Scalar field $\phi : (M_n, g_{\mu\nu}) \to \mathbb{R}$

$$Z[g_{\mu\nu}] = \int d\phi \, e^{-\int \partial_\mu \phi \, \partial^\mu \phi \, \mathrm{vol}(g)}$$

$$= \frac{1}{\sqrt{\det{}' \Delta}}$$

is an interesting function of the metric.

Freed + Hopkins [1301.5959] formalize a notion of background fields as a "sheaf on Man_n" a functor

$$\mathcal{F}: \mathrm{Man}_n^{op} \longrightarrow \mathrm{Set} \quad \left(\begin{array}{c}\text{better,}\\ \text{simplicial}\\ \text{sets}\end{array}\right)$$

We then put $\mathcal{F}$-structures on our bordisms to define an enhanced category $\mathrm{Bord}_{\langle n-1,n\rangle}(\mathcal{F})$ and we can define a TFT with such background fields as

$$F: \mathrm{Bord}_{\langle n-1,n\rangle}(\mathcal{F}) \longrightarrow \mathrm{VECT}$$

Works well for discrete structures like orientation, principal G-bundles with finite G. Much more needs to be said if $\mathcal{F}$ includes, say, Riemannian metrics, conformal structures, ...

Interesting open problem: Formulate field theories where $\mathcal{F}$ includes foliations.

11. A Survey Of Some Famous Examples Of TFT's

One of the most famous examples is 3d Chern-Simons theory. Perhaps the simplest example is constructed from a gauge theory with a $U(1)$ gauge field, i.e. a connection on a principal $U(1)$ bundle $P \longrightarrow M_3$, with M_3 a 3-dimensional oriented manifold. Locally the gauge field is described by a real 1-form A with globally-defined fieldstrength $F \in \Omega^2(M_3)$. Locally $F = dA$

The exponentiated action in the path integral is:

$$\exp\left(i \frac{k}{4\pi} \int_{M_3} A\, dA \right)$$

Where we have normalized A so that F has periods in $2\pi\mathbb{Z}$.

The action doesn't look gauge invariant, but under "small" gauge trms $A \to A + d\epsilon$

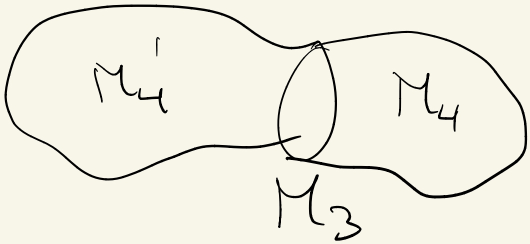
$$A\,dA \to A\,dA + d(\epsilon\, dA)$$

So if M_3 is compact w/out boundary $\int_{M_3} A\,dA$ is gauge invariant. But under large gauge trans $A \to A + \omega$ $\omega \in \Omega^1_{\mathbb{Z}}$ it is still not quite well-defined. Better way:

One can prove that it is possible to extend $P \to M_3$ and its connection to $P \to M_4$ with $\partial M_4 = M_3$. Then the identity $F \wedge F = d(A\,dA)$ and Stokes' theorem motivates the hypothetical definition:

$$\int_{M_3} A\,dA \overset{?}{=} \int_{M_4} F \wedge F$$

Problem: The extension is not unique.



and in general $\int_{M_4} F \wedge F \neq \int_{M_4'} F \wedge F$

What saves the day is that on the closed manifold $M_4 \cup M_4'$ $\int F \wedge F \in (2\pi)^2 \mathbb{Z}$

$$\therefore \quad \frac{1}{2\pi} \int_{M_4} F \wedge F \in \mathbb{R} / 2\pi \mathbb{Z}$$

is well-defined.

$\therefore$ if k is an even integer

$$\exp\left(i \frac{k}{4\pi} \int_{M_4} F \wedge F \right)$$ is a

good action principle

(If we endow M_3, M_4 with spin structures we can extend to k an odd integer. The inclusion of spin structures is a good example of the inclusion of background nondynamical fields $\mathcal{F}$.)

Note that the path integral "measure"

$$\exp\left(i \frac{k}{4\pi} \int A\, dA\right)$$

is metric independent. So we expect this to define a topological field theory. That is almost true but in defining the path integral

$$F(M_3) = \int_{\mathcal{A}/\mathcal{G}} [dA]\, e^{i\frac{k}{4\pi}\int_{M_3} A\,dA}$$

one must introduce a metric to define one loop determinants. One finds an overall dependence on the metric

$$F(M_3) = e^{2\pi i \frac{c}{24} \omega_{cs}(g)} \times \underbrace{\tilde{F}(M_3)}_{\text{metric independent}}$$

(with $c = k$)

There are various approaches to deal with the metric anomaly.

1. Try to subtract off the gravitational Chern-Simons term as a "local counterterm." (Witten's paper on the Jones polynomial does this.)

2. Include background fields so the TFT is defined on the correct bordism category $\mathrm{Bord}_{\langle 2,3\rangle}(\mathcal{F})$.

An example of an $\mathcal{F}$ is a framing, but a cruder structure known as a 2-framing will suffice (see Atiyah's paper on TFT for a definition of 2-framing).

This basic example of 3d U(1) Chern-Simons Theory can be generalized in several ways:

A.) Many U(1) fields A^I $I=1,\dots,r$

$$\text{Action} = \frac{1}{4\pi}\int K_{IJ}\, A^I dA^J$$

very useful in the QHE.

The matrix K_{IJ} must be a symmetric integral matrix and it determines an integral lattice. The quantum amplitudes can be expressed in terms of invariants of this lattice.

B) Nonabelian gauge fields.

Now take a connection on a nonabelian principal G-bundle $P \to M_3$. for G a compact simple group. Locally the connection is $d + A$ $\quad A \in \Omega^1(U_\alpha, \mathfrak{g})$ $\mathfrak{g} = Lie(G)$. We can form the Chern-Simons form

$$d\, Tr\left(AdA + \frac{2}{3}A^3\right) = Tr(F \wedge F)$$

and the Chern-Simons action

$$\frac{k}{4\pi}\int_{M_3} Tr\left(AdA + \frac{2}{3}A^3\right) = \frac{k}{4\pi}\int Tr\, F \wedge F$$

For a suitable notion of trace (e.g. $Tr = Tr_N$ for $G = SU(N)$)

k must be an integer, and then (for M_3 cpt w/out bdry)

$$\exp\left(i \frac{k}{4\pi} \int_{M_3} Tr\left(A dA + \frac{2}{3} A^3\right)\right)$$

$$= \exp\left(i \frac{k}{4\pi} \int_{M_4} Tr\, F \wedge F \right)$$

is well-defined.

Exercise: Compute the change of $Tr\left(A dA + \frac{2}{3} A^3\right)$ under gauge trm

$$d+A \longrightarrow g^{-1}(d+A)g$$

to see why the 3-form

$\mathrm{Tr}\left(AdA+\frac{2}{3}A^3\right)$ is not globally well-defined on M_3.

One can define a nice Chern–Simons–Witten TFT for <u>any</u> compact group. In general it is determined simply by a choice of

G — compact Lie group

$k \in H^4(BG, \mathbb{Z})$ "level"

This topic continues to influence much current research

For more on 3d Chern-Simons Theory see my 2019 TASI lectures and the many references Therein.

One can also extend 3d CS theory to noncompact groups. Here the flavor is quite different and they are typically <u>not</u> TFT's. For example, state spaces are typically ∞-dim'l.

For more about this very active research topic see:

1. Witten "Analytic Continuation Of Chern-Simons
2. Tudor Dimofte - review
3. Andersen + Kashaev - ICM Address

C.) Chern-Simons theories

can be "upgraded" to higher-dimensi. theories by using new exterior data: For example for a suitably normalized closed 1-form we can contemplate a 4d theory like

$$\int \omega \, \mathrm{Tr}\left(A dA + \frac{2}{3} A^3\right)$$

One needs to be careful here to get a well-defined propagator. This kind of theory was studied by Losev, Moore, Nekrasov, Shatashvili c. 1995 and further developed in Nikita Nekrasov's PhD thesis.

More recently it has played a major role in works by Kevin Costello and callaborators, especially in providing new insights into integrable models. See, e.g. the series of papers of Costello, Yamazaki, and Witten.

A similar expression makes an appearance in an effective action for 4d topological insulators with nontrivial first Chern-class of the band structure bundle. See G. Moore, "A Comment on Berry Connections."

D. "BF Theories"

Another way to generalize to higher dimensions is to replace the closed 2-form F of Maxwell thery by an ℓ-form

$$F \in \Omega^\ell(M_n)$$

$$dF = 0 \implies F = dA$$

A: locally defined $(\ell-1)$-form.

If we have A, A' then

$$\exp\left(ik \int_{M_n} A\, dA'\right)$$

makes sense so long as

$$l + l' = n+1$$

A good way to think about these actions makes use of differential cohomology — discussed later.

Nonexamples:

A. 2D Yang-Mills

$$\text{Action} = \int \operatorname{tr}(\phi F) + \underset{\substack{\uparrow \\ \text{area form}}}{\mu} \operatorname{tr}(\phi^2)$$

$$F(\Sigma_g, A) = \sum_{R:\text{ irrep}} (\dim R)^{2-2g} e^{-A C_2(R)}$$

does not have good $A \to 0$ for $g=0$

B. Donaldson-Witten, Vafa-Witten, Kapustin-Witten, Rozansky-Witten, Gromov-Witten, X-Floer, Y-Floer, Z-Floer

Q-closed sector w/ $Q^2=0$

Very different feeling

Typically Only partially defined